Sylow Subgraphs in Self-Complementary Vertex Transitive Graphs

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Abstract: A graph is self-complementary if it is isomorphic to its complement. A graph is vertex transitive if for each choice of vertices u and v there is an automorphism that carries the vertex u to v. The number of vertices in a self-complementary vertex transitive graph must necessarily be congruent to 1 mod 4. However, Muzychuk has shown that if p^m is the largest power of a prime p dividing the order of a self-complementary vertex transitive graph, then p^m must individually be congruent to 1 mod 4. This is accomplished by establishing the existence of a self-complementary vertex transitive subgraph of order p^m, a result reminiscent of the Sylow theorems. This article is a self-contained survey, culminating with a detailed proof of Muzychuk's result.

Keywords: graph theory, vertex transitive, self-complementary

1 Introduction

A graph whose group of automorphisms acts transitively on the vertex set is called vertex transitive. If the complement of the edge set of a graph forms a graph isomorphic to the original, then the graph is said to be self-complementary. The combination of these two conditions severely restricts the possibilities for such graphs. For example, there are $165\,091\,172\,592$ non-isomorphic graphs on 12 vertices [19], with 720 of these being self-complementary [18], 74 are vertex transitive [19], but none are both self-complementary and vertex transitive (as Theorem 3.1 implies). On 13 vertices, there are $50\,502\,031\,367\,952$ graphs [19], of which $5\,600$ are self-complementary [18], 14 are vertex transitive [19], yet only 2 are simultaneously self-complementary and vertex transitive [20].

There are various ways to describe graphs that have much symmetry, and it is these highly symmetric graphs that occur in many contexts, such as error-correcting codes, combinatorial designs, and simple groups [23]. Conditions that guarantee symmetry can be purely combinatorial (regular, distance regular) or based on the automorphism group of the graph (vertex transitive, symmetric, distance transitive). Here we consider graphs that have what is perhaps the simplest form of symmetry based on the automorphism group: being vertex transitive. However, being

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self-complementary is also a way to describe another type of symmetry but with more of a combinatorial flavor (though a permutation is involved). This article concentrates on elementary number-theoretic conditions that limit the possibilities for graphs that have both flavors of symmetry. Along the way we present a result of Muzychuk that goes further and describes the existence of subgraphs of self-complementary vertex transitive graphs that are again self-complementary and vertex transitive.

Vertex transitive graphs have been studied extensively. We mention three examples illustrating why self-complementary graphs are of interest. First, the problem of determining if two graphs are isomorphic is in NP, but is not known if it is in P or if it is NP-complete. Colbourn and Colbourn [3] were able to show that the graph isomorphism problem is polynomially reducible to the problem of recognizing if a single graph is self-complementary. Second, the Ramsey number $R(m, n)$ is the least integer so that any graph of that order, or greater, is guaranteed to contain a complete subgraph on m vertices, or its complement contains a complete subgraph on n vertices. A diagonal Ramsey number has $m = n$. Self-complementary vertex transitive graphs can be useful for bounding diagonal Ramsey numbers [1, 2, 13]. For example, Luo, *et.al.* [13] employed self-complementary graphs to prove that $R(19, 19) \geq 17885$. In the extreme, the "best" graphs for this purpose are also vertex transitive. Third, Lovász [12] showed that the information-theoretic Shannon capacity of a self-complementary vertex transitive graph of order n is $\sqrt{n}$.

We are interested here in the possibilities for the orders of self-complementary vertex transitive graphs. Assume for the remainder of this introduction that p and q are distinct primes. It is straightforward to show using graph-theoretic techniques (Theorem 3.1) that a self-complementary vertex transitive graph must have order congruent to 1 modulo 4 [17, 22]. A graph is *circulant* if its automorphism group has a permutation composed of a single cycle that moves every vertex of the graph, and this condition implies vertex transitivity. Early work on self-complementary graphs assumed this more restrictive notion of symmetry. In 1962 Sachs [22] showed that if a circulant self-complementary graph has order p^{2s}, then $p \equiv 1 \pmod 4$. Further, if the order is pq, then both $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$. Interest in self-complementary graphs was rekindled in 1985 with the appearance of papers by Rao [16] and Suprunenko [21]. In 1996 Fronček, Rosa, and Širáň [4] extended Sach's results to show that if p is any prime divisor of the order of a self-complementary circulant graph, then $p \equiv 1 \pmod 4$. At about the same time, Koolen [6] in 1997 relaxed the assumption of being circulant and used combinatorial techniques to show that there are no self-complementary vertex transitive graphs of order $3p$. Concurrently, Li [7] employed the heavy machinery of the classification of the finite simple groups to extend Sach's and Koolen's results to show that when the order of a self-complementary vertex transitive graph is pq, then both $p \equiv 1 \pmod 4$ and $q \equiv 1 \pmod 4$. Li's survey [8] provides more information about these results and the status of various other problems related to self-complementary vertex transitive graphs through 1998.

More recently, Muzychuk [14] has generalized all of the above results by showing that if p^m is the highest power of p dividing the order of a self-complementary vertex transitive graph, then $p^m \equiv 1 \pmod 4$ (Theorem 5.2). The main device in this proof is to use the Sylow theorems (all of them!) to establish the existence of self-complementary vertex transitive subgraphs of order p^m (Theorem 5.1). Therefore it is natural to refer to these subgraphs as "Sylow subgraphs." Moreover, this result is the best possible, since for every integer n with the property that for each prime divisor p the highest power of p dividing n is congruent to 1 modulo 4, there exists a self-complementary vertex transitive graph of order n (Theorem 4.5). Here we provide the

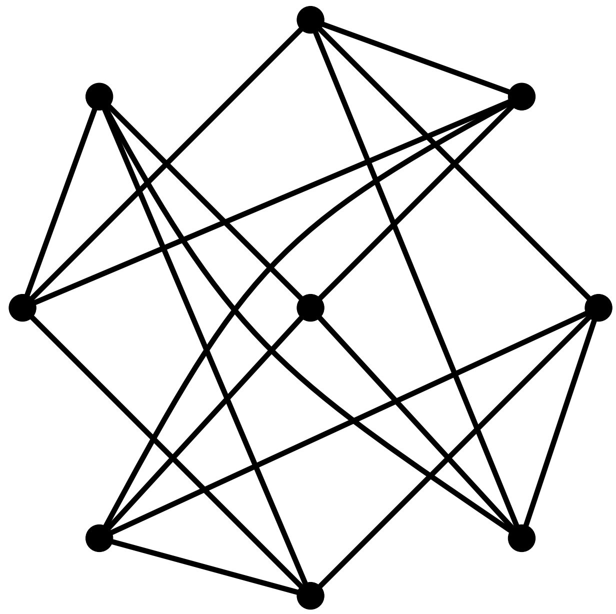

Fig. 1 A self-complementary vertex transitive graph on 9 vertices.

necessary preliminary results, and finish with the construction of these Sylow subgraphs and the subsequent complete characterization of the orders of self-complementary vertex transitive graphs.

2 Definitions and Notation

A *graph* $\Gamma = (V, E)$ is a finite set of *vertices* V, together with a set of *edges* E that is composed of two-element subsets of V. A permutation σ of V is an *automorphism* of Γ if it preserves edges, that is whenever $\{u, v\} \in E$, then $\{\sigma(u), \sigma(v)\} \in E$. The set of all automorphisms of a graph Γ forms a group, $\mathbf{Aut}(\boldsymbol{\Gamma})$. A *complementing map* of Γ is a permutation σ of V such that $\{u, v\} \in E$ if and only if $\{\sigma(u), \sigma(v)\} \notin E$. For example, the graph of Figure 1 is self-complementary, as can be seen by the existence of the complementing map obtained by rotating the figure about the central vertex through an angle of $\pi/4$. We denote the set of all complementing maps of a graph Γ as $C(\Gamma)$. We can now carefully state our two main definitions.

Definition 2.1 *A graph $\Gamma = (V, E)$ is vertex transitive if its automorphism group acts transitively on the set of vertices. That is, for any $u, v \in V$ there exists $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\sigma(u) = v$.*

Definition 2.2 *A graph $\Gamma = (V, E)$ is self-complementary if $C(\Gamma)$ is non-empty. That is, there is at least one complementing map of Γ.*

Figure 1 is an example of a construction from Section 4 that yields graphs that are both self-complementary and vertex transitive.

The complement of a set X will be denoted by $\overline{X}$. Many of the other definitions and constructions used here are also described in [5].

3 Two Basic Properties

Theorem 3.1 ([17, 22]) *If $\Gamma = (V, E)$ is a self-complementary regular graph with $|V| = n$, then $n \equiv 1 \pmod 4$.*

Proof Let r denote the common degree of the vertices of Γ. The complement, $\overline{\Gamma}$, is also a regular graph, with common degree $n - r - 1$. Since Γ is self-complementary, these degrees must be equal, yielding $n = 2r + 1$, and so n is odd.

There are a total of $\binom{n}{2} = \frac{n(n-1)}{2}$ possible edges among the vertices V. In order for Γ and $\overline{\Gamma}$ to be isomorphic, this set must partition into two sets of equal size, E and $\overline{E}$. Thus $n(n-1)$ is divisible by 4, and since n is odd, we have the desired conclusion. $\square$

The main goal of this paper is to show how this simple result extends to the individual prime power divisors of n.

Lemma 3.2 ([22, 17, 21]) *Suppose $\Gamma = (V, E)$ is a graph with $|V| \equiv 1 \pmod 4$ and $\sigma \in C(\Gamma)$. Then σ fixes exactly one element of V, and the remaining cycle lengths of σ are divisible by 4.*

Proof First suppose that σ fixes two elements of V, say u and v. Then $\sigma(\{u, v\}) = \{\sigma(u), \sigma(v)\} = \{u, v\}$. The pair $\{u, v\}$ is either an edge of Γ or $\overline{\Gamma}$, and either case contradicts that σ is a complementing map. Thus σ fixes at most one element of V.

Now suppose that σ contains a transposition, a cycle of length two. Considering the two vertices of this transposition as a pair, and proceeding as above, it is clear this is also impossible.

Suppose that m is any odd integer and define $\tau = \sigma^m$. If we apply the permutation σ^2 to E, it first maps E to $\overline{E}$ (relative to the set of all possible two-element subsets of V), and then maps $\overline{E}$ back to E. Since τ is an odd power of σ, $\tau(E) = \overline{E}$, and therefore $\tau \in C(\Gamma)$. Now the previous properties observed for σ are equally true for τ, so τ fixes at most one element and has no transpositions.

Suppose that σ contains a non-identity cycle ρ of length m. Then τ would contain m fixed points, with $m > 1$, which is a contradiction. Further, suppose that σ contains a cycle ρ of length $2m$. Then τ would contain transpositions, also a contradiction. Therefore, non-identity cycle lengths in σ are divisible by 4. $\square$

Remark Suppose a graph Γ is self-complementary and vertex transitive. Since Γ is vertex transitive, it is regular, so by Theorem 3.1 $|V| \equiv 1 \pmod 4$. Since Γ is self-complementary, every $\sigma \in C(\Gamma)$ fixes exactly one vertex by Lemma 3.2. The lone vertex left fixed by complementing maps of self-complementary vertex transitive graphs plays a pivotal role in the proofs that follow.

4 Constructing Self-Complementary Vertex Transitive Graphs

In this section we describe a construction of self-complementary vertex transitive graphs due to Rao, that shows that in a certain sense our final result is the best possible. The results in this section can be found in Rao [16].

Definition 4.1 *Suppose that* $\Gamma_1 = (V_1,\, E_1)$ *and* $\Gamma_2 = (V_2,\, E_2)$ *are graphs. Define the lexicographic product,* $\Gamma_1\,[\Gamma_2]$*, by* $V(\Gamma_1\,[\Gamma_2]) = V_1 \times V_2$*, the usual Cartesian product, and* $E(\Gamma_1\,[\Gamma_2]) = \{\, \{(u_1,u_2),(v_1,v_2)\} \mid \{u_1,v_1\} \in E_1,\ \text{or}\ u_1 = v_1,\ \{u_2,v_2\} \in E_2\}$.

The next two propositions are routine to verify.

Theorem 4.2 *If* Γ_1 *and* Γ_2 *are vertex transitive graphs, then the lexicographic product* $\Gamma_1\,[\Gamma_2]$ *is also vertex transitive.*

Proof Use the fact that if σ_1 and σ_2 are automorphisms of Γ_1 and Γ_2, respectively, then the function ψ defined by $\psi(u_1,u_2) = (\sigma_1(u_1),\, \sigma_2(u_2))$ is a automorphism of $\Gamma_1\,[\Gamma_2]$. □

Theorem 4.3 *If* Γ_1 *and* Γ_2 *are self-complementary graphs, then the lexicographic product* $\Gamma_1\,[\Gamma_2]$ *is also self-complementary.*

Proof First, note that $\overline{\Gamma_1\,[\Gamma_2]} = \overline{\Gamma_1}\,\left[\overline{\Gamma_2}\right]$. Then, for complementing maps, σ_1, σ_2 of Γ_1, Γ_2, respectively, construct as above the complementing map ψ of $\Gamma_1\,[\Gamma_2]$. □

With these two general results about lexicographic products, we extend the building blocks of the following construction to create a variety of self-complementary vertex transitive graphs.

Theorem 4.4 *Let* p *be a prime such that* $p^r \equiv 1 \pmod 4$*. Then there exists a self-complementary vertex transitive graph with* p^r *vertices.*

Proof Write $p^r = 4k+1$ and let x be a primitive element of the finite field $\mathrm{GF}(p^r)$. That is, x is a generator of the multiplicative group of non-zero elements, $\mathrm{GF}(p^r)^*$. Define a graph $\Gamma = (V,\, E)$ by $V = \{0, x, x^2, \ldots,\, x^{4k} = 1\}$, $E = \{\, \{u,v\} \mid u - v = x^{2s},\ s \in \mathbb{Z}\,\}$.

To verify that Γ is self-complementary, check that $\sigma = (0)(x\,x^2\,x^3\,\ldots\,x^{4k})$ is a complementing map.

To verify that Γ is vertex transitive, suppose that u and v are two vertices of Γ, and check that the permutation ρ defined on V by $\rho(w) = w + (v-u)$ is an automorphism of Γ that carries u to v. □

The graphs of this construction are known as Paley graphs, and are also examples of a more general construction known as Cayley graphs, which employ groups as vertex sets. The graph of Figure 1 is a Paley graph on 9 vertices, where x is a root of the irreducible polynomial x^2+2x+2 over GF(3). The central vertex is 0, and powers of x are placed around the perimeter clockwise, beginning with x at the top center vertex. The complementing map σ is a clockwise rotation through an angle of $\pi/4$.

Theorem 4.5 ([16], Theorem 4.5) *Suppose* $n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$*, where the* p_i *are distinct primes and each* $p_i^{r_i} \equiv 1 \pmod 4$*. Then there exists a self-complementary vertex transitive graph with* n *vertices.*

Proof Construct Paley graphs of order $p_i^{r_i}$ as described in Theorem 4.4, combine them with lexicographic products, and apply Theorems 4.2 and 4.3 to see that the result is a self-complementary vertex transitive graph of the desired order. $\square$

Next we show that this description of the prime decomposition of the order of a self-complementary vertex transitive graph is not only sufficient, but necessary.

5 Sylow Subgraphs

Our main result is the following, due to Muzychuk [14], whose proof we follow closely.

Theorem 5.1 ([14], Theorem 2) *Suppose that $\Gamma = (V, E)$ is a self-complementary vertex transitive graph with $|V| = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$, where the p_i are distinct primes. Then for each $1 \leq i \leq k$, Γ contains a vertex-induced subgraph of order $p_i^{r_i}$ that is both self-complementary and vertex transitive.*

Proof Let p be one of the prime divisors of $|V|$ and suppose p^m is the highest power of p that divides $|V|$. Let G denote the automorphism group of Γ and suppose p^n is the highest power of p that divides $|G|$. For a subgroup H and a group element σ, we will denote conjugation by $H^\sigma = \sigma H \sigma^{-1}$.

For a vertex v of Γ, denote its stabilizer in G by $G_v = \{\sigma \in G \mid \sigma(v) = v\}$ and its orbit by $v^G = \{\sigma(v) \mid \sigma \in G\}$. Since Γ is vertex transitive, $|G| = |G_v||v^G| = |G_v||V|$ for any v, and therefore p^{n-m} is the highest power of p that divides the order of G_v Thus, a Sylow p-subgroup of G_v has order p^{n-m}, while a Sylow p-subgroup of G has order p^n.

Denote the unique fixed point of $\sigma \in C(\Gamma)$ by $x(\sigma)$, whose existence is guaranteed by Lemma 3.2. For any p-group, P of G, define

$$D(P) = \{\sigma \in C(\Gamma) \mid P^\sigma = P, \, |P_{x(\sigma)}| = p^{n-m}\}.$$

Our goal is to find a Sylow p-subgroup of G, P, for which $D(P)$ is non-empty. To this end, define $\mathcal{P} = \{P \mid P \text{ is a } p\text{−group in } G, \, D(P) \neq \varnothing\}$.

First, we show that $\mathcal{P}$ is not empty. Since Γ is self-complementary, choose an arbitrary $\sigma \in C(\Gamma)$. Let $x = x(\sigma)$ denote its fixed point. By Sylow's First Theorem, choose a Sylow p-subgroup of G_x, say P. Since σ normalizes G_x, P^σ is also a Sylow p-subgroup of G_x. Therefore, by Sylow's Third Theorem, P and P^σ are conjugate in G_x so there exists $\tau \in G_x$ such that $P^\sigma = P^\tau$.

Define $\widetilde{\sigma} = \tau^{-1}\sigma$. Note that $\widetilde{\sigma}$ is a complementing map since σ is a complementing map and τ^{-1} is an automorphism. Then P satisfies the first condition for membership in $\mathcal{P}$ since $P^{\widetilde{\sigma}} = P$. Because τ is in the stabilizer of x, and σ fixes x by definition, $\widetilde{\sigma}$ also fixes x. Lemma 3.2 says this fixed point is unique, so $x(\widetilde{\sigma}) = x = x(\sigma)$. Thus $P_{x(\widetilde{\sigma})} = P_{x(\sigma)} = P_x = P$, where the last equality follows since P is a subgroup of G_x. Therefore $|P_{x(\widetilde{\sigma})}| = |P| = p^{n-m}$. This is the second condition required of P, and hence $\mathcal{P} \neq \varnothing$.

Now we can choose an element M from $\mathcal{P}$ that is maximal with respect to inclusion. We will determine that M is a Sylow p-subgroup of G and use it to build a self-complementary vertex transitive subgraph. Let $K = N_G(M)$ be the normalizer of M in G. By Sylow's Second Theorem, we can choose a Sylow p-subgroup of K that contains M, which we denote as $\widetilde{M}$. We now show that $\widetilde{M} \in \mathcal{P}$, implying that $M = \widetilde{M}$.

Since $D(M) \neq \varnothing$, choose $\sigma \in D(M)$, and let $x = x(\sigma)$. Thus, $M^\sigma = M$ and $|M_x| = p^{n-m}$. Because σ normalizes M, it also normalizes K. This makes $(\widetilde{M})^\sigma$ also a Sylow p-subgroup of K. Hence $\widetilde{M}$ and $(\widetilde{M})^\sigma$ are conjugate in K which implies there exists $\kappa \in K$ such that $(\widetilde{M})^\sigma = (\widetilde{M})^\kappa$.

Define $\widetilde{\sigma} = \kappa^{-1}\sigma$. Note first that $\widetilde{\sigma}$ is a complementing map and that $(\widetilde{M})^{\widetilde{\sigma}} = \widetilde{M}$, meeting the first condition for $D(\widetilde{M})$ to be non-empty. Let $W = x^K$ be the orbit of the fixed point of σ under the action of K. Because σ fixes x, and σ normalizes K, we must have W invariant under σ. That is, $\sigma(W) = \{\sigma(w) \mid w \in W\} = W$. Trivially, $x \in W$ and by Lemma 3.2, σ has non-identity cycles with lengths divisible by 4, and so $|W| \equiv 1 \pmod 4$.

Now, $\widetilde{\sigma}(W) = \kappa^{-1}\sigma(W) = \kappa^{-1}(W) = W$. So $|\widetilde{\sigma}(W)| \equiv 1 \pmod 4$. Combining this with the cycle structure of $\widetilde{\sigma}$, it follows that the unique fixed point of $\widetilde{\sigma}$ must be in W also. With $x = x(\sigma)$ and $x(\widetilde{\sigma})$ both in $W = x^K$, there exists $\rho \in K$ such that $\rho(x(\sigma)) = x(\widetilde{\sigma})$.

One easily checks that elements of ${M_x}^\rho$ fix $x(\widetilde{\sigma})$. Moreover, since ρ normalizes M, we have ${M_x}^\rho \subseteq M^\rho = M$. Therefore, ${M_x}^\rho \subseteq M_{x(\widetilde{\sigma})}$ and we have

$$p^{n-m} = |M_x| = |{M_x}^\rho| \leq |M_{x(\widetilde{\sigma})}| \leq |\widetilde{M}_{x(\widetilde{\sigma})}| \leq p^{n-m}.$$

In particular, $|\widetilde{M}_{x(\widetilde{\sigma})}| = p^{n-m}$, establishing the second condition for $D(\widetilde{M})$ to be non-empty. Thus $\widetilde{M} \in \mathcal{P}$.

The subgroup M was chosen as a maximal element of $\mathcal{P}$, yet $M \subseteq \widetilde{M}$, so $M = \widetilde{M}$. Thus M is a Sylow p-subgroup of K. This means that M is also a Sylow p-subgroup of G, as we now establish by contradiction. Suppose that M is not a Sylow p-subgroup of G, so because it is a p-group, it must be properly contained in some Sylow p-subgroup of G, say P^*. Because M and P^* are p-groups, $M \subset N(M)$ and there exists $g \in P^* \setminus M$ such that g normalizes M. Consider the subgroup generated by g and M, $H = \langle M, g\rangle$. First, $H \subseteq K$. Second, H is a subgroup of P^*, so H is a p-group. Third, M is a proper subset of H. Thus, $M \subset H \subseteq K$, and the existence of H contradicts that M is a Sylow p-subgroup of K. Thus we conclude that M is a Sylow p-subgroup of G and therefore M has order p^n.

We now use properties of M to create the desired subgraph of Γ. Since $D(M)$ is non-empty, choose a $\sigma \in D(M)$. Then $\sigma \in C(\Gamma)$, $M^\sigma = M$, $|M_{x(\sigma)}| = p^{n-m}$. Let $X = x(\sigma)^M$, the orbit of the fixed point of σ under the action of M, and consider the subgraph, Δ, of Γ induced by the vertices in X.

We now show that the subgraph Δ has the properties required by the conclusion of the theorem. Since σ normalizes M and fixes $x(\sigma)$, $\sigma(X) = X$, making σ a permutation of the vertex set of Δ. Because $\sigma \in C(\Gamma)$, σ is a complementing map of Δ, and therefore Δ is a self-complementary graph. The set X is an orbit under the action of $M \subseteq G$, so the elements of M are automorphisms of Δ that act transitively on the vertices of Δ, and hence Δ is a vertex transitive graph. Finally, in the action of the group M on X, we have $p^n = |M| = |M_{x(\sigma)}||x(\sigma)^M| = p^{n-m}|X|$ implying that $|X| = p^m$ and therefore Δ is a graph of order p^m. $\square$

Theorem 5.2 ([14], Theorem 1) *Suppose that Γ is a self-complementary vertex transitive graph with $|V| = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$, where the p_i are distinct primes. Then for each $1 \leq i \leq k$, $p_i^{r_i} \equiv 1 \pmod 4$.*

Proof Apply Theorem 3.1 to the Sylow subgraphs determined in Theorem 5.1. $\square$

6 Recent Progress on Characterizing Self-Complementary Vertex Transitive Graphs

Suppose that H is a subset of a group G, with the properties that $1 \notin H$, and whenever $a \in H$, then $a^{-1} \in H$. The Cayley graph determined by G and H has G as its vertex set, and the pair $\{a,\, b\}$ is an edge if and only if $a^{-1}b \in H$. Then it is easy to see that G is a vertex transitive graph since each group element gives rise to a natural automorphism of the graph. The Paley graphs described in Theorem 4.5 are examples of Cayley graphs. Until recently, every known example of a self-complementary vertex transitive graph was also a Cayley graph. Li [7, Question 3.2] asked if there were self-complementary vertex transitive graphs that are not Cayley graphs. Together with Praeger, he answers the question in the affirmative by finding an infinite family of self-complementary vertex transitive graphs that are not Cayley graphs [9].

A stronger form of symmetry goes by the name of symmetric (equivalently, 1-arc-transitive). Suppose for any two ordered pairs (u, v) and (x, w) that are also edges of a graph ($\{u, v\}$, $\{w, x\} \in E(\Gamma)$), there exists an automorphism $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\sigma(u) = x$ and $\sigma(v) = w$. Then Γ is said to be *symmetric*. Preisert [15] has recently used results about primitive permutation groups to completely classify the symmetric self-complementary graphs as the Paley graphs, a second infinite parameterized family, and just one other graph not present in either family.

In [10, 11] Li and Praeger have generalized the notion of a self-complementary vertex transitive graph by considering homogeneous factorizations of complete graphs. To define a *homogeneous factorization* of the complete graph K_n, begin with a partition of the edge set into k sets, E_i, $1 \le i \le k$, thereby inducing k subgraphs, Γ_i, $1 \le i \le k$. Then require that there be a group of permutations, G, from S_n that leaves the partition invariant, permutes the parts E_i transitively, and induces a vertex transitive automorphism group on each Γ_i. It should be clear that the case when $k = 2$ is exactly that of self-complementary vertex transitive graphs. Initial results in special cases suggest that number-theoretic conditions similar to those described here might be possible for homogeneous factorizations. For example, in the special case of a *cyclic* homogeneous factorization of K_n with k parts and with p^m as the highest power of the prime p dividing n, then

$$p^m \equiv \begin{cases} 1 \pmod{2k} & \text{if } p \text{ is odd} \\ 1 \pmod{k} & \text{if } p = 2. \end{cases}$$

Furthermore, for each combination of k and n meeting this condition, there is a construction that yields a cyclic homogeneous factorization of K_n with k parts.

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